- Let formula *A* contain propositional variables $p_1, p_2, \ldots, p_n$, find the principal disjunctive normal form (**PDNF**) of *A*.

(1) **Find a Disjunctive Normal Form (DNF):** $A' = B_1 \vee B_2 \vee \ldots \vee B_s$, where each B_j is a **simple conjunction** , $j=1,2, \ldots ,s$.

(2) If a term B_j lacks either p_i, or $\neg p_i$, B_j expand it using

$$B_j \Leftrightarrow B_j \wedge (p_i \vee \neg p_i) \Leftrightarrow (B_j \wedge p_i) \vee (B_j \wedge \neg p_i)$$

Repeat until every conjunction becomes a **minterm** (length n).

(3) **Remove Duplicate Minterms:** Replace repeated minterms $m_i \vee m_i$ with m_i.

(4) **Order Minterms:** Arrange the minterms in ascending order of their indices.

↳ Find PCNF

- Let formula A contain propositional variables $p_1,p_2,\ldots,p_n$, find the principal **Conjunctive** normal form (**PCNF**) of A.

(1) **Find a Conjunctive Normal Form (CNF)** $A'=B_1\wedge B_2\wedge \ldots \wedge B_s$, where each B_j is a **simple disjunction**, $j=1,2, \ldots ,s$.

(2) If a term B_j lacks either p_i or $\neg p_i$ for some variable p_i, expand it using: $B_j \Leftrightarrow B_j\vee(p_i\wedge\neg p_i) \Leftrightarrow (B_j\vee p_i)\wedge(B_j\vee\neg p_i)$

Repeat until every disjunction becomes a **maxterm** (length n).

(3) Replace repeated maxterms (e.g., $M_i \wedge M_i$) with a single M_i.

(4) Arrange the maxterms in ascending order of their indices (e.g., M_0, M_1, M_2, …).

e.g. Example: Find the PDNF and PCNF of $\neg(p\to q) \vee \neg r$

■ Step 1: Find PDNF (Minterms)

$\neg(p\to q)\vee\neg r \Leftrightarrow (p\wedge\neg q)\vee\neg r$ (Implication equivalence, De Morgan)

$$p\wedge\neg q \Leftrightarrow (p\wedge\neg q)\wedge 1$$
$$\Leftrightarrow (p\wedge\neg q)\wedge(\neg r\vee r)$$
$$\Leftrightarrow (p\wedge\neg q\wedge\neg r)\vee(p\wedge\neg q\wedge r)$$
$$\Leftrightarrow m_4\vee m_5$$

$$\neg r \Leftrightarrow (\neg p\vee p)\wedge(\neg q\vee q)\wedge\neg r$$
$$\Leftrightarrow (\neg p\wedge\neg q\wedge\neg r)\vee(\neg p\wedge q\wedge\neg r)\vee(p\wedge\neg q\wedge\neg r)\vee(p\wedge q\wedge\neg r)$$
$$\Leftrightarrow m_0\vee m_2\vee m_4\vee m_6$$

$$\neg(p\to q)\vee\neg r \Leftrightarrow m_0\vee m_2\vee m_4\vee m_5\vee m_6$$
$$\Leftrightarrow \Sigma(0,2,4,5,6)$$

- Step 2: Find PCNF (Maxterms)

$$\neg(p\to q)\vee\neg r \Leftrightarrow (p\vee\neg r)\wedge(\neg q\vee\neg r)$$

$$p\vee\neg r \Leftrightarrow p\vee 0\vee\neg r$$

$$\Leftrightarrow p\vee(q\wedge\neg q)\vee\neg r$$

$$\Leftrightarrow (p\vee q\vee\neg r)\wedge(p\vee\neg q\vee\neg r)$$

$$\Leftrightarrow M_1\wedge M_3$$

$$\neg q\vee\neg r \Leftrightarrow (p\wedge\neg p)\vee\neg q\vee\neg r$$

$$\Leftrightarrow (p\vee\neg q\vee\neg r)\wedge(\neg p\vee\neg q\vee\neg r)$$

$$\Leftrightarrow M_3\wedge M_7$$

$$\neg(p\to q)\vee\neg r \Leftrightarrow M_1\wedge M_3\wedge M_7$$

$$\Leftrightarrow \Pi(1,3,7)$$

- A ***quick way*** to obtain the PDNF (or PCNF) of a formula A with n propositional variables is to complete the missing variables to form all possible minterms (or maxterms).
- A **conjunctive clause** (simple AND term) of length k can be expanded into 2^{n-k} minterms.

Such as: Formulation ***p,q,r***

$$q \Leftrightarrow (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee q \vee r)$$

$$\Leftrightarrow m_2 \vee m_3 \vee m_6 \vee m_7$$

- **A disjunctive clause** (simple OR term) of length *k* can be expanded into 2^{n-k} maxterms.
- **Such as:** $p \vee \neg r \Leftrightarrow (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$

$$\Leftrightarrow M_1 \wedge M_3$$

e.g. **Example:** Find the PDNF (Principal Disjunctive Normal Form)

$$A \Leftrightarrow (\neg p \wedge q) \vee (\neg p \wedge \neg q \wedge r) \vee r$$

■**Solution (Quick Method):**

$$\neg p \wedge q \Leftrightarrow (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \Leftrightarrow m_2 \vee m_3$$

$$\neg p \wedge \neg q \wedge r \Leftrightarrow m_1$$

$$r \Leftrightarrow (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$$

$$\Leftrightarrow m_1 \vee m_3 \vee m_5 \vee m_7$$

$$A \Leftrightarrow m_1 \vee m_2 \vee m_3 \vee m_5 \vee m_7 \Leftrightarrow \Sigma(1,2,3,5,7)$$

e.g. **Example:** Find the PCNF (Principal Conjunctive Normal Form)

$$B \Leftrightarrow \neg p \wedge (p \vee q \vee \neg r)$$

Solution (Quick Method):

$$\neg p \Leftrightarrow (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

$$\Leftrightarrow M_4 \wedge M_5 \wedge M_6 \wedge M_7$$

$$p \vee q \vee \neg r \Leftrightarrow M_1$$

Get $B \Leftrightarrow M_1 \wedge M_4 \wedge M_5 \wedge M_6 \wedge M_7 \Leftrightarrow \Pi(1,4,5,6,7)$

- Use PDNF to Determine the ***True Assignments*** and ***False Assignments*** of a Formula.
- Let formula A contain n propositional variables. If the PDNF (Principal Disjunctive Normal Form) of A has s minterms, then:
  - ***True Assignments***: s assignments corresponding to the binary representations of the minterm indices.
  - ***False Assignments***: 2^{n-s} assignments not covered by the minterms.

e.g. **Example:** $\neg(p\rightarrow q)\vee\neg r \Leftrightarrow m_0\vee m_2\vee m_4 \vee m_5 \vee m_6$

True Assignments: 000,010,100,101,110;

False Assignments: 001,011,111

- Use PDNF to Determine the type of a Formula.
- Let *A* be a formula with ***n*** propositional variables.
  - *A* is a ***tautology* if and only if** its PDNF (Principal Disjunctive Normal Form) contains **all 2^n minterms.**
  - *A* is a ***contradiction* if and only if** its PDNF contains **no minterms** (denoted as 0).
  - *A* is ***satisfiable* if and only if** its PDNF contains **at least one minterm.**

e.g. **Example:** Determining formula types using principal disjunctive normal form (PDNF).

- **Problem:** Classify the following formulas:

  (1) $A \Leftrightarrow \neg(p \rightarrow q) \wedge q$ (2) $B \Leftrightarrow p \rightarrow (p \vee q)$ (3) $C \Leftrightarrow (p \vee q) \rightarrow r$

- **Solution:**

  (1) $A \Leftrightarrow \neg(\neg p \vee q) \wedge q \Leftrightarrow (p \wedge \neg q) \wedge q \Leftrightarrow 0$ *contradiction*

  (2) $B \Leftrightarrow \neg p \vee (p \vee q) \Leftrightarrow 1 \Leftrightarrow m_0 \vee m_1 \vee m_2 \vee m_3$ *tautology*

  (3) $C \Leftrightarrow \neg(p \vee q) \vee r \Leftrightarrow (\neg p \wedge \neg q) \vee r$

  $\Leftrightarrow (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

  $\vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$

  $\Leftrightarrow m_0 \vee m_1 \vee m_3 \vee m_5 \vee m_7$ *Non-tautological satisfiable formula*

- Use PDNF to determine whether two formulas are logically equivalent.

e.g. **Example:** Use principal disjunctive normal form (PDNF) to determine if the following pair of formulas are equivalent:

(1) p and $(\neg p \vee q) \rightarrow (p \wedge q)$

Solve: $p \Leftrightarrow p \wedge (\neg q \vee q) \Leftrightarrow (p \wedge \neg q) \vee (p \wedge q) \Leftrightarrow m_2 \vee m_3$

$(\neg p \vee q) \rightarrow (p \wedge q) \Leftrightarrow \neg(\neg p \vee q) \vee (p \wedge q)$

$\Leftrightarrow (p \wedge \neg q) \vee (p \wedge q) \Leftrightarrow m_2 \vee m_3$

Then: $p \Leftrightarrow (\neg p \vee q) \rightarrow (p \wedge q)$

e.g. **Example:** Use principal disjunctive normal form (PDNF) to determine if the following pair of formulas are equivalent:

(2) $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$

Solve: $(p \wedge q) \vee r \Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$

$\vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$

$\Leftrightarrow m_1 \vee m_3 \vee m_5 \vee m_6 \vee m_7$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$\Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$

$\Leftrightarrow m_5 \vee m_6 \vee m_7$

Then: $(p \wedge q) \vee r$ is not equal to $p \wedge (q \vee r)$

e.g. **Example:** A company needs to select personnel from A, B, and C for an overseas assignment, subject to the following conditions: (1) If A goes, then C must go; (2) If B goes, then C cannot go; (3) Exactly one of A or B must go.

■ **Question:** How many possible selection plans are there?

■ **Solution:**

- Define propositions: ***p***: Send A ; ***q***: Send B; ***r***: Send C

  (1) $p \to r$, (2) $q \to \neg r$, (3) $(p \wedge \neg q) \vee (\neg p \wedge q)$

- Combine into a logical formula:

  $$C = (p \to r) \wedge (q \to \neg r) \wedge ((p \wedge \neg q) \vee (\neg p \wedge q))$$

↳ Use PDNF to designate the personnel selection plan(cont.)

- Find principal disjunctive normal form of *C*

$$C=(p\to r)\land(q\to\neg r)\land((p\land\neg q)\lor(\neg p\land q))$$

$$\Leftrightarrow (\neg p\lor r)\land(\neg q\lor\neg r)\land((p\land\neg q)\lor(\neg p\land q))$$

$$\Leftrightarrow ((\neg p\land\neg q)\lor(\neg p\land\neg r)\lor(r\land\neg q)\lor(r\land\neg r))$$
$$\land((p\land\neg q)\lor(\neg p\land q))$$

$$\Leftrightarrow ((\neg p\land\neg q)\land(p\land\neg q))\lor((\neg p\land\neg r)\land(p\land\neg q))$$
$$\lor((r\land\neg q)\land(p\land\neg q))\lor((\neg p\land\neg q)\land(\neg p\land q))$$
$$\lor((\neg p\land\neg r)\land(\neg p\land q))\lor((r\land\neg q)\land(\neg p\land q))$$

$$\Leftrightarrow (p\land\neg q\land r)\lor(\neg p\land q\land\neg r)$$

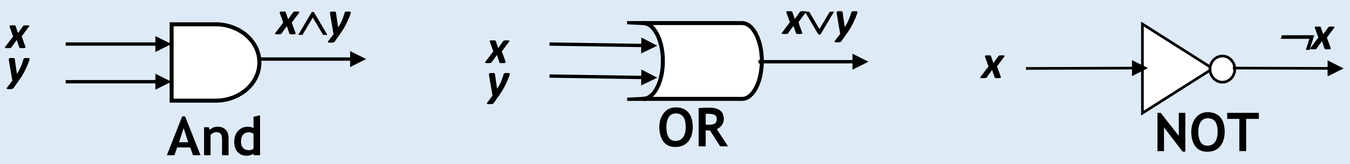
- **True Assignment :101,010**

Plan 1: Send A and C

Plan 2: Send B

Use PDNF to design a control circuit

e.g. **Example:** A lamp is controlled by two switches. Pressing either switch can turn the lamp on or off. Use the gate circuits in the diagram to design the control circuit.



Solution:

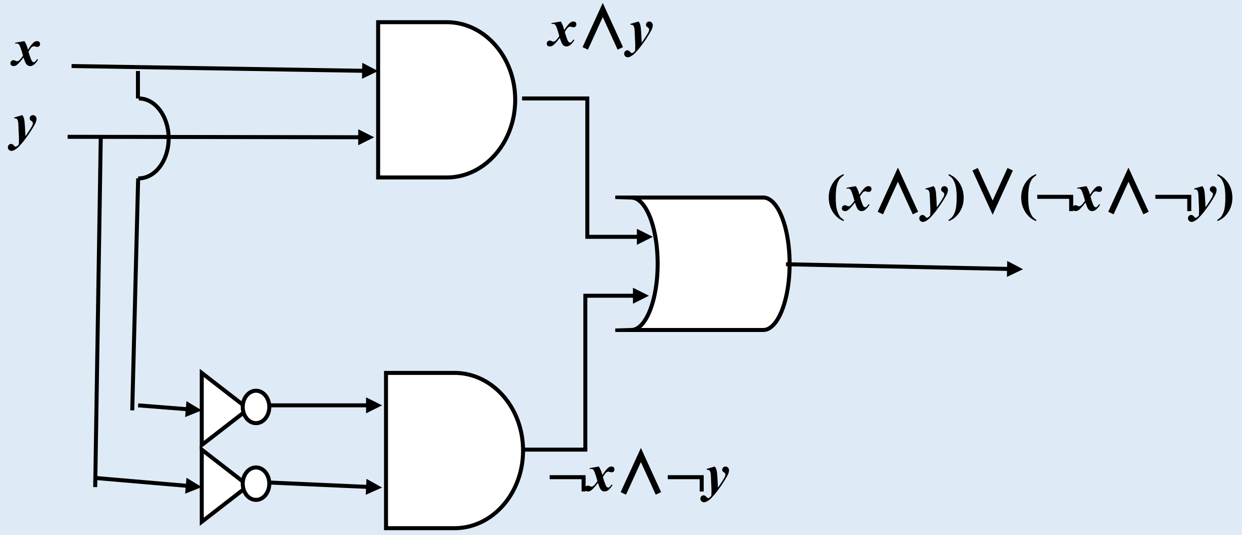
- Let ***x*** and ***y*** represent the states of the two switches (0 = off, 1 = on).
- Let ***F*** represent the lamp's state (1 = on, 0 = off).
- Assume the lamp is initially on (F=1) when x=y=0.
- **Canonical DNF** (Principal Disjunctive Normal Form) **of *F***

$$F=m_0\wedge m_3$$

$$=(\neg x\wedge\neg y)\vee(x\wedge y)$$

x	y	F
0	0	1
0	1	0
1	0	0
1	1	1

↳ Use PDNF to design a control circuit (cont.)



If the initial condition is set such that the light turns on (***F***=1) only when exactly one of ***x*** or ***y*** is '1', how would the circuit behave?

Let: $A \Leftrightarrow m_{i_1} \vee m_{i_2} \vee \cdots \vee m_{i_s}$

Non exited Minterm is $m_{j_1}, m_{j_2}, \cdots, m_{j_t}$, So $t=2^{n-s}$,

Thus:

$$\neg A \Leftrightarrow m_{j_1} \vee m_{j_2} \vee \cdots \vee m_{j_t}$$

$$A \Leftrightarrow \neg(m_{j_1} \vee m_{j_2} \vee \cdots \vee m_{j_t})$$

$$\Leftrightarrow \neg m_{j_1} \wedge \neg m_{j_2} \wedge \cdots \wedge \neg m_{j_t}$$

$$\Leftrightarrow M_{j_1} \wedge M_{j_2} \wedge \cdots \wedge M_{j_t}$$

Objective：

Key Concepts：